Int. J. Solids Structures Vol. 20, No. 9/10, pp. 897–909, 1984
Printed in the U.S.A.

0020-7683/84 $3.00 + .00
Pergamon Press Ltd.

DYNAMIC NECKING OF AN ELASTIC–PLASTIC CYLINDER UNDER UNIAXIAL STRETCHING

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(*Received* 7 *June* 1983; *in revised form* 7 *November* 1983)

Abstract—The axisymmetric necking process of an elastic–plastic cylinder under uniaxial stretching is analyzed by treating the problem as an initial-value-eigenvalue problem. One end of the cylinder is subjected to a prescribed constant axial velocity relative to the other end and both ends are shear free. The eigenmodes of deviated motion departing from a uniform stretch beyond that of quasi-static bifurcation by Hutchinson and Miles are determined. Of all uniformly excited eigenmodes of motion, only certain modes will grow. The rate of necking depends on the geometry, material properties and stretching velocity of the cylinder.

1. INTRODUCTION

The phenomenon of localized necking in an elastic–plastic body under tensile loading has been observed and studied by many investigators[1]. Most of the necking problems analyzed are based on Hill's quasi-static theory of bifurcation and uniqueness[2]. In a quasi-static sense, the problem of the onset of necking in an elastic–plastic cylinder under uniaxial tension has been analyzed[3–5]. Hutchinson and Miles[5] have shown that the state of uniform uniaxial tension is unique prior to the maximum support load of the specimen and that the true stress at bifurcation is greater than that at the maximum load by an amount which depends on the geometry and material properties.

Dynamically, the necking process of the cylinder is influenced by an additional factor of stretching rate. There have been relatively few attempts to solve the dynamic problem. An approach based on a one-dimensional formulation[6] has been suggested for determining the dynamic axisymmetric necking process. In this paper, the axisymmetric necking process of an incompressible elastic–plastic cylinder under uniaxial stretching is further analyzed by a two-dimensional formulation. The ends of the cylinder are stretched in such a way that the ends remain free of tangential tractions and the lateral surface traction free. One end of the cylinder is subjected to a prescribed constant axial velocity relative to the other end. The axial velocity is relatively small in comparing with the elastic axial wave speed of the cylinder. For continuity, the state of quasi-static bifurcation as determined by Hutchinson and Miles[5] is employed as a reference state. The dynamic necking problem is solved by employing a finite strain, dynamic, quasi-bifurcation theory[7, 8] which is briefly described in the paper. Numerical results for a number of cylinders are presented and discussed.

2. QUASI-BIFURCATION

A certain motion of a system of n degrees of freedom is considered stable if, after a sufficiently small disturbance, the system remains to follow the undisturbed motion. In other words, the undisturbed motion is stable if the deviated motion, $\zeta_r(t)$, $r = 1, \ldots, n$, the

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difference between the disturbed and undisturbed motions in n generalized coordinates, remains small.

For any deviated motion, stable or not, there is the following identity[8]:

$$\frac{1}{2}\zeta_r(t)\zeta_r(t) = \frac{1}{2}\gamma_r\gamma_r + \int_0^t \left[\dot{\gamma}_r\gamma_r + \int_0^{t'} \dot{\zeta}_r\dot{\zeta}_r \, d\tau + \int_0^{t'} \ddot{\zeta}_r\zeta_r \, d\tau \right] dt' \tag{1}$$

in which dots indicate differentiation with respect to time t and γ_r and $\dot{\gamma}_r$ are the initial values of the deviated motion. When the generalized forces, P_r, of the system are configuration-dependent and when the deviated motion is relatively small, the deviated motion is governed by the following variational equations[7]:

$$\ddot{\zeta}_r = A_{rs}\zeta_s = \Delta P_r, \quad r = 1, \ldots, n; \quad s = 1, \ldots, n \tag{2}$$

where the coefficients A_{rs} are known functions of time depending on the undisturbed motion only. In eqn (1), without loss of generality, the initial condition may be taken as $\gamma_r = 0$ at a chosen time $t = 0$. Consequently, eqn (1) shows clearly that the undisturbed motion is unstable or the deviated motion grows monotonously when the quadratic term

$$Q = \Delta P_r\zeta_r = \ddot{\zeta}_r\zeta_r = A_{rs}\zeta_r\zeta_s \tag{3}$$

is positive in a time period for any possible deviated motion. Furthermore, a variation of Q yields the following relationship:

$$\delta Q = 2A_{rs}\zeta_s\delta\zeta_r = 2\ddot{\zeta}_r\delta\zeta_r \tag{4}$$

which may be employed, by appropriate coordinate transformation, to obtain equations of motion in other generalized coordinates.

The quadratic form Q at a given time can always be reduced to a linear combination of squares[9] such as:

$$Q = \alpha_m\eta_m^2, \quad m = 1, \ldots, n \tag{5}$$

where $\alpha_m(t)$ are the time-dependent eigenvalues, η_m are related to ζ_r by an orthogonal transformation

$$\zeta_r = l_{mr}\eta_m, \quad r = 1, \ldots, \nu \tag{6}$$

where l_{mr} are the directional numbers of n mutually orthogonal vectors, η_m, in the ζ_r space known as eigenvectors or as eigenmode of motion. Such transformation is called a transformation to principal axes. In general, the directional numbers l_{mr} may be time dependent[8]. For cases where l_{mr} are always constants (or nearly constant), a variation of Q leads to the uncoupled ordinary differential equations

$$\ddot{\eta}_m = \alpha_{\underline{m}}\eta_{\underline{m}}, \quad m = 1, \ldots, n \tag{7}$$

where the underscores are placed under the indices to suspend the summation convention. If ζ_r and η_r are normalized, Q assumes the greatest value equal to the largest eigenvalue, α_k, corresponding to the configuration η_k in the ζ_r space by a theorem of Weierstrasse[9]. Therefore, α_k and η_k may also be determined by the extremum condition. If $\alpha_k > 0$ and $\alpha_k > \alpha_r$ for $r \neq k$, then the eigenmode of motion, η_k, grows at the highest rate, provided that the initial disturbance has such a component. Such phenomenon is called a quasi-bifurcation phenomenon. The above concept and approach are applied to the solution of the present problem.

Of an elastic–plastic solid of a volume, V, let the undisturbed motion be described by the displacement $U_K(X_M, t)$ in the X_M Cartesian coordinate system. The deviated (or additional) displacement is denoted by $u_K(X_M, t)$. The corresponding Lagrangian strains are e_{KL} and ϵ_{KL} and the corresponding Piola–Kirchhoff stresses are S_{KL} and s_{KL}, respectively. When u_k is relatively small, the additional strain, ϵ_{KL}, is given by

$$\epsilon_{KL} = \frac{1}{2}(u_{K,L} + u_{L,K} + U_{M,K}u_{M,L} + U_{M,L}u_{M,K}) \tag{8}$$

and the deviated motion is governed by the following equation of motion:

$$\Delta P_M = [S_{KL}u_{M,L} + s_{KL}(\delta_{ML} + U_{M,L})]_{,K} = \rho\ddot{u}_M \tag{9}$$

where ρ is the initial mass density. The deviated motion satisfies the following boundary conditions: $u_K(X_M, t) = 0$ on that part of the boundary with prescribed kinematic conditions and

$$[S_{KL}u_{M,L} + s_{KL}(\delta_{ML} + U_{M,L})]N_K = \Delta T_M \tag{10}$$

on that part of the boundary with prescribed surface traction T_M, where N_K is the outward unit normal to the surface. The quadratic term for the deviated motion of the elastic–plastic continuum is given by

$$Q(u_k) = \int_V \Delta P_M u_M \, dV = \int_V \rho\ddot{u}_M u_M \, dV. \tag{11}$$

For a body which has no change in surface traction and no interior discontinuities of the variables, the quadratic term may be simplified as

$$Q(u_K) = -\int_V (S_{KL}u_{M,L} + s_{KL}\epsilon_{KL}) \, dV. \tag{12}$$

The motion and its stability of such solid may be determined by analyzing the functional $Q(u_K)$.

By employing an appropriate constitutive relation, the function Q can be expressed in terms of the deviated displacement field u_M. When the deviated motion is relatively small, the disturbed and undisturbed motions have nearly identical histories such that s_{KL} appear as small deviations from the stress path of S_{KL} in either loading or unloading condition. Thus, the stress deviations may be expressed in terms of the strain deviations by a general expression such as

$$s_{KL} = \begin{cases} C_{KLMN}\epsilon_{MN} & \text{for} \quad B_{KL}\epsilon_{KL} < 0 \\ C_{KLMN}\epsilon_{MN} - \dfrac{1}{g}B_{KL}B_{MN}\epsilon_{MN} & \text{for} \quad B_{MN}\epsilon_{MN} > 0. \end{cases} \tag{13}$$

Here, C_{KLMN} is the tensor of instantaneous elastic moduli, B_{KL} is the unit tensor normal to the elastic domain in the strain-increment space, and g is a positive scalar which determines the strain hardening of the material and depends upon the strain history of the undisturbed motion. It is noted that the quasi-static bifurcation is characterized by the occurrence of a non-zero solution, u_M, to the variational equation $\delta Q = 0$ as shown by Hill[2].

3. AXIAL MOTION

Consider a solid circular cylinder having homogeneous elastic–plastic properties. One end of the cylinder is subjected to a constant axial velocity v relative to the other end. It is assumed that the axial velocity is low enough such that the axial stress waves have reflected many times throughout the length of the rod and that its dynamic state is the nearly uniform state of uniaxial tension at a time prior to the beginning of the necking process. It is also assumed that the ends are stretched in such a way that the ends remain free of tangential traction and the lateral surface traction free. Consider that there exist initial axisymmetric imperfections in the cylinder which are infinitesimal in magnitudes such that they produce negligible disturbances to the nearly uniform state of uniaxial tension prior to the bifurcation process. The imperfections may lead to an initial state of the deviated motion. The onset of quasi-static bifurcation in the cylinder, which has been determined by Hutchinson and Miles[5], occurs at a true axial stress σ_c. This state of stress is employed as a reference state which occurs at a time $t = 0$ when the length and radius of the cylinder are L and R, resp.

If initial imperfections or disturbances are absent, the dynamic state of the cylinder at $t > 0$ is the nearly uniform state of axial tension with a true axial stress σ and a length of $(L + vt)$. This state is considered as the undisturbed state. Assuming the material is incompressible, the second Piola–Kirchoff axial stress, S_z, may be expressed in terms of the true axial stress as

$$S_z = \frac{\sigma}{\left(1 + \frac{v}{c}\tau\right)^2} \tag{14}$$

where

$$c = \sqrt{\frac{E}{\rho}} \quad \text{and} \quad \tau = \frac{ct}{L}. \tag{15}$$

Here, E is the Young's modulus of elasticity and c the elastic wave speed. The true axial stress may be expressed in terms of τ by employing a Ramberg–Osgood relation[10] between the true stress σ and natural strain e such as

$$\frac{e}{e_y} = \frac{\sigma}{\sigma_y} + \left(\frac{3}{7}\right)\left(\frac{\sigma}{\sigma_y}\right)^{\bar{n}} \tag{16}$$

where e_y and $\sigma_y = Ee_y$ are the effective yield strain and yield stress and $\bar{n}$ is the hardening exponent. Let E_t be the tangent modulus for an uniaxial increment of true stress according to

$$E_t = \frac{d\sigma}{de}. \tag{17}$$

The derivative of the tangent modulus at the reference state is found to be

$$\left.\frac{dE_t}{d\sigma}\right|_{\sigma=\sigma_c} \cong -\frac{E_t^c}{\sigma_c}(\bar{n} - 1)\left(1 - \frac{E_t^c}{E}\right) \tag{18}$$

where E_t^c is the value of E_t at σ_c. Retaining only the first term of the Taylor's series expansion of E_t in the neighborhood of σ_c, it is found that

$$E_t \cong E_t^c - (\bar{n} - 1)\frac{E_t^c}{\sigma_c}\left(1 - \frac{E_t^c}{E}\right)(\sigma - \sigma_c). \tag{19}$$

Following the definitions of E_t and the additional natural strain

$$e = \frac{v\tau}{c\left(1 + \frac{v\tau}{c}\right)}, \tag{20}$$

the integration of (19) yields

$$\sigma \cong \sigma_c + \frac{\sigma_c}{(\bar{n}-1)(1-\lambda_1)}\left[1-\left(1+\frac{v}{c}\tau\right)^{-\lambda_2(\bar{n}-1)(1-\lambda_1)}\right] \quad \text{for } \tau \geqslant 0 \tag{21}$$

where

$$\lambda_1 = \frac{E_t^c}{E}, \quad \lambda_2 = \frac{E_t^c}{\sigma_c}.$$

4. DEVIATED MOTION

When the undisturbed motion is stable, all the disturbed motions will remain in the neighborhood of the undisturbed motion. When unstable, only certain deviated motions, however small in magnitude and whatever way excited initially, will grow and depart from the neighborhood. The axisymmetric, deviated motions, of the cylinder may be described in terms of the deviated displacements (u_r, u_z) in a cylindrical coordinate system (r, θ, z) with $0 \leqslant r \leqslant R$ and $0 \leqslant z \leqslant L$ associated with the reference state. When the deviated motions are relatively small, the constitutive relations of an incompressible elastic–plastic solid having isotropic elastic properties and under continued plastic loading may be written as [5]

$$\begin{aligned} s_z &= 2G\epsilon_z - \frac{1}{g}\epsilon_z + p \\ s_r &= 2G\epsilon_r + \frac{1}{2g}\epsilon_z + p \\ s_\theta &= 2G\epsilon_\theta + \frac{1}{2g}\epsilon_z + p \\ s_{rz} &= 2G\epsilon_{rz} \end{aligned} \tag{22}$$

where G is the elastic shear modulus and $p = \frac{1}{3}(s_r + s_\theta + s_z)$. Here and in the remainder of the paper, the physical components of the deviated Piola–Kirchhoff stresses $(s_r, s_\theta, s_z, s_{rz})$ and deviated Lagrangian strains $(\epsilon_r, \epsilon_\theta, \epsilon_z, \epsilon_{rz})$ are used. For an incompressible material, $\epsilon_r + \epsilon_\theta + \epsilon_z = 0$ and $3G = E$. From the definition of E_t and (22), it follows that [5]

$$\frac{1}{g} = 2\left[G + \frac{1}{3}\left(2\sigma - E_t\right)\right]. \tag{23}$$

The equations of deviated motion and associated boundary conditions may be written as

$$\begin{aligned} \frac{1}{r}\frac{\partial(rs_r)}{\partial r} + \frac{\partial s_{rz}}{\partial z} - \frac{1}{r}s_\theta + \sigma\frac{\partial^2 u_r}{\partial z^2} &= \rho\frac{\partial^2 u_r}{\partial t^2} \\ \frac{\partial s_z}{\partial z} + \frac{1}{r}\frac{\partial(rs_{rz})}{\partial z} + \sigma\frac{\partial^2 u_z}{\partial z^2} &= \rho\frac{\partial^2 u_z}{\partial t^2} \end{aligned} \tag{24}$$

$$\left.\begin{aligned} s_r &= 0 \\ s_{rz} &= 0 \end{aligned}\right\} \text{for } r = R \quad \text{and} \quad \left.\begin{aligned} u_z &= 0 \\ s_{rz} &= 0 \end{aligned}\right\} \text{for } z = 0, L. \tag{25}$$

The equations of motion may be expressed in terms of the deviated displacement components by using the constitutive equations (22) and the following relationships between strains and displacements:

$$\epsilon_r = \frac{\partial u_r}{\partial r}, \; \epsilon_\theta = \frac{u_r}{r}, \; \epsilon_z = \frac{\partial u_z}{\partial z}, \; \epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right). \tag{26}$$

Following the approach by Hutchinson and Miles[5], a function $\Phi(r, z, t)$, which ensures the condition of incompressibility, is introduced with

$$u_r = -\frac{\partial \Phi}{\partial z} \quad \text{and} \quad u_z = \frac{1}{r}\frac{\partial (r\Phi)}{\partial r}. \tag{27}$$

Furthermore, the function Φ can be written in a separated form

$$\Phi = R^2\phi(r, t)\sin\frac{k\pi z}{L}, \quad k = 1, 2, 3, \ldots \tag{28}$$

such that the boundary conditions on $z = 0$, L are satisfied. Using eqns (22)–(28), the traction-free boundary conditions (25) on the lateral surface can be expressed as

$$\left.\begin{aligned} L(\phi) + \gamma^2\phi = 0 \\ \gamma^2\frac{E_t - \sigma}{G}\frac{\partial(\zeta\phi)}{\partial\zeta} - \frac{\partial}{\partial\zeta}[\zeta L(\phi)] - 2\gamma^2\phi = 0 \end{aligned}\right\} \text{on } \zeta = 1 \tag{29}$$

where

$$\gamma = \frac{k\pi R}{L}, \quad \zeta = \frac{r}{R}. \tag{30}$$

The operator in (29) is defined by

$$L(\phi) = \frac{\partial}{\partial\zeta}\left[\zeta^{-1}\frac{\partial(\zeta\phi)}{\partial\zeta}\right]. \tag{31}$$

It may be shown by further manipulation that eqns (29) may be reduced to a single constraint equation.

$$(E_t - \sigma + G)\zeta\frac{\partial\phi}{\partial\zeta} + (E_t - \sigma - G)\phi = 0 \text{ on } \zeta = 1. \tag{32}$$

The equations of deviated motion (24) may be further reduced to a single hyperbolic equation in terms of the function ϕ subject to the constraint by eqn (32). Instead of solving this equation directly, the following variational approach is employed.

5. EIGENVECTORS

The quadratic functional Q, by (12), for this case of axisymmetric deformation, may be specialized to the form

$$Q(z) = -2\pi\int_0^R\int_0^L\left\{S_z\left[\left(\frac{\partial u_r}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right] + (s_r\varepsilon_r + s_z\varepsilon_z + s_\theta\varepsilon_\theta + 2s_{rz}\varepsilon_{rz})\right\} r\,\mathrm{d}r\,\mathrm{d}z \tag{33}$$

Employing eqns (22), (26)–(28) and integrating (33) with respect to z, we find

$$Q = -\pi R^2 L \int_0^1 \left\{ \left(2G - \frac{3}{2g}\right)\gamma^2\left(\frac{\phi}{\zeta} + \frac{\partial\phi}{\partial\zeta}\right)^2 + 2G\gamma^2\left[\left(\frac{\partial\phi}{\partial\zeta}\right)^2 + \left(\frac{\phi}{\zeta}\right)^2\right] \right.$$
$$+ G\left(\phi\gamma^2 - \frac{\phi}{\zeta^2} + \frac{1}{\zeta}\frac{\partial\phi}{\partial\zeta} + \frac{\partial^2\phi}{\partial\zeta^2}\right)^2$$
$$\left. + \gamma^2\left[\gamma^2\phi^2 + \left(\frac{\phi}{\zeta} + \frac{\partial\phi}{\partial\zeta}\right)^2\right]\left[\frac{1 - \left(1 + \frac{v}{c}\tau\right)^{-\lambda_2(\bar{n}-1)(1-\lambda_1)}}{(\bar{n}-1)(1-\lambda_1)} + 1\right] \cdot \frac{\sigma_c}{\left(1 + \frac{v}{c}\tau\right)^2} \right\} \zeta \, d\zeta. \tag{34}$$

For small values of γ, the solution of ϕ may be represented by the following power series[5]

$$\phi(\zeta, \tau) = \sum_{n=1}^{N} b_n(\tau) \frac{(-1)^{n-1}\left(\frac{\gamma}{2}\right)^{2n-1}}{(n-1)!n!} \zeta^{2n-1}. \tag{35}$$

The dimensionless amplitudes $b_n(\tau)$ are subjected to the traction free boundary condition (32), or

$$\sum_{n=1}^{N} b_n(\tau) \frac{(-1)^{n-1}\left(\frac{1}{2}\gamma\right)^{2n-1}}{(n-1)!n!} [(E_t - \sigma)n + G(n-1)] = 0. \tag{36}$$

The eigenvalues and eigenvectors of the functional Q may be determined by the extremum condition. Substituting (35) into (34) and integrating with respect to ζ, the functional Q is reduced to a quadratic form in terms of $b_n(\tau)$. By introducing Lagrange's multiplier b_{N+1}, the extremum condition and the condition of constraint may be combined as

$$\delta Q + \delta\left\{2\pi R L b_{N+1} \sum_{n=1}^{N} b_n \frac{(-1)^{n-1}\left(\frac{1}{2}\gamma\right)^{2n-1}}{(n-1)!n!} [n(E_t - \sigma) + G(n-1)]\right\} = 0. \tag{37}$$

For a chosen value of τ, the variation of (37) with respect to each parameter $b_n(\tau)$ yields

$$\sum_{n=1}^{N+1} C_{mn} b_n(\tau) = 0, \quad m = 1, \ldots, N+1. \tag{38}$$

The eigenvalues, α_p, may be determined by the characteristic equation

$$|C_{mn} - \alpha_p \delta_{mn}| = 0. \tag{39}$$

The corresponding eigenvectors may be obtained by solving the simultaneous equations

$$(C_{mn} - \alpha_p \delta_{mn}) b_n = 0, \quad m = 1, \ldots, N+1, p = 1, \ldots, N+1. \tag{40}$$

The eigenvectors may be expressed by the orthogonal transformation

$$b_n(\tau) = l_{mn} \eta_m(\tau). \tag{41}$$

For a case of deviated motions from a uniform unperturbed axial motion, the directional numbers l_{mn} of the eigenvectors η_m are practically constants. Substitution of (41) into (35) yields

$$\phi(\zeta, \tau) = \sum_{m=1}^{N+1} \sum_{n=1}^{N+1} \frac{(-1)^{n-1}\left(\frac{\gamma}{2}\right)^{2n-1}}{(n-1)!n!} \zeta^{2n-1} l_{mn} \eta_m. \tag{42}$$

Substituting (42) into (34) and observing the orthogonal properties of l_{mn}, the functional Q may be expressed in terms of the eigenmodes of motion as

$$Q = -\sum_p \pi R^2 LE\alpha_p \eta_p^2 \tag{43}$$

where

$$\begin{aligned}
\alpha_p = \int_0^1 \Bigg\{ & \left(2\lambda_3 - \frac{3}{2\lambda_g}\right)\gamma^2 \left(\sum_{n=1}^{N+1} \frac{(-1)^{n-1}\left(\frac{\gamma}{2}\right)^{2n-1}}{(n-1)!n!} 2n\zeta^{2n-2} l_{pn}\right)^2 \\
& + 2\lambda_3\gamma^2 \left[\left(\sum_{n=1}^{N+1} \frac{(-1)^{n-1}\left(\frac{\gamma}{2}\right)^{2n-1}}{(n-1)!n!} (2n-1)\zeta^{2n-2} l_{pn}\right)^2 \right. \\
& \left. + \left(\sum_{n=1}^{N+1} \frac{(-1)^{n-1}\left(\frac{\gamma}{2}\right)^{2n-1}}{(n-1)!n!} \zeta^{2n-2} l_{pn}\right)^2\right] \\
& + \lambda_3 \left[\sum_{n=1}^{N+1} \frac{(-1)^{n-1}\left(\frac{\gamma}{2}\right)^{2n-1}}{(n-1)!n!} \zeta^{2n-3}(\gamma^2\zeta^2 + 4n^2 - 4n) l_{pn}\right]^2 \\
& + \gamma^2 \left[\gamma^2 \left(\sum_{n=1}^{N+1} \frac{(-1)^{n-1}\left(\frac{\gamma}{2}\right)^{2n-1}}{(n-1)!n!} \zeta^{2n-1} l_{pn}\right)^2 + \left(\sum_{n=1}^{N+1} \frac{(-1)^{n-1}\left(\frac{\gamma}{2}\right)^{2n-1}}{(n-1)!n!} 2n\zeta^{2n-2} l_{pn}\right)^2\right] \\
& \times \left[\frac{1 - \left(1 + \frac{v}{c}\tau\right)^{-\lambda_2(\bar{n}-1)(1-\lambda_1)}}{(\bar{n}-1)(1-\lambda_1)} + 1\right] \frac{\lambda_c}{\left(1 + \frac{v\tau}{c}\right)^2} \Bigg\} \zeta \, d\zeta
\end{aligned} \tag{44}$$

where $\lambda_3 = \frac{G}{E}$, $\lambda_g = \frac{g}{E}$, $\lambda_c = \frac{\sigma_c}{E}$.

6. NECKING PROCESS

The functional Q may also be represented by the integral at the right side of (11), or

$$Q = \int_V \rho(\ddot{u}_r u_r + \ddot{u}_z u_z) \, dV. \tag{45}$$

By substituting eqns (27), (28), (35) and (41) into (45) and integrating, it is found that

$$Q = \pi\rho R^4 L^{-1} c^2 \sum_p \beta_p \frac{d^2\eta_p}{d\tau^2}\eta_p \tag{46}$$

where

$$\beta_p = \int_0^1 \left\{ \left(\sum_{n=1}^{N=1} \frac{(-1)^{n-1}\left(\frac{\gamma}{2}\right)^{2n-1}}{(n-1)!n!} \zeta^{2n-1}\gamma l_{pn} \right)^2 + \left(\sum_{n=1}^{N+1} \frac{(-1)^{n-1}\left(\frac{\gamma}{2}\right)^{2n-1}}{(n-1)!n!} \zeta^{2n-2} 2n l_{pn} \right)^2 \right\} \zeta \, d\zeta. \tag{47}$$

A variation of the functional Q with respect to the pth eigenmode of motion leads to the following uncoupled equations of motion

$$\frac{d^2\eta_p}{d\tau^2} = \Lambda_p(\tau)\eta_p \quad p = 1, \ldots, N+1 \tag{48}$$

where

$$\Lambda_p(\tau) = -2\left(\frac{L}{R}\right)^2 \frac{\alpha_p}{\beta_p}. \tag{49}$$

Equations (48) describe an axisymmetric necking process accompanying with the axial stretching of a cylinder. The development of the necking process depends on the function $\Lambda_p(\tau)$. If $\Lambda_p < 0$, the pth mode of motion is oscillatory. If $\Lambda_p > 0$ in a time interval, the pth mode of motion grows monotonously in amplitude in the time interval. When the value of $\Lambda_p(\tau)$ of a certain mode of motion becomes positive at the earliest time ($\tau = 0$), it marks the beginning of the necking process. For that particular mode of deformation, the condition $\Lambda_p(0) = 0$ or $\alpha_p(0) = 0$, which corresponds to the quasi-static bifurcation criterion, leads to the determination of the reference state having a true axial stress of σ_c. For an elastic–plastic solid having isotropic elastic properties, the value of σ_c as given by Hutchinson and Miles[5] may be reduced to that by the following expression

$$\frac{E_t^c - \sigma_c}{G} - \frac{\gamma^2}{8}\left[\frac{E_t^c - \sigma_c}{G} - \frac{\sigma_c}{G} - \left(\frac{E_t^c - \sigma_c}{G}\right)^2\right] + \frac{\gamma^4}{192}\left[1 + \frac{E_t^c - \sigma_c}{G}\left(3 + \frac{4\sigma_c}{G}\right) - 2\left(\frac{E_t^c - \sigma_c}{G}\right)^2 + \left(\frac{E_t^c - \sigma_c}{G}\right)^3 - \frac{\sigma_c}{G}\right] = 0. \tag{50}$$

It has been shown by Hutchinson and Miles[5] that the quasi-static bifurcation cannot take place prior to the maximum tensile load. Denoting the tangent modulus at the maximum load point by E_t^m, the maximum load condition yields he corresponding true stress $\sigma_m = E_t^m$. Furthermore, $\sigma_c > \sigma_m$, $E_t^c < E_t^m$ and $E_t^c < \sigma_c$ for a material having properties described by eqns (16)–(19). In a dynamic process of stretching the cylinder to a state having a true stress σ greater than σ_c, as shown by eqn (21), the corresponding E_t is such that the value of $(\sigma - E_t)$ increases as the stretching progresses beyond σ_c. The value of Λ_p of a certain eigenmode of motion increases positively as $(\sigma - E_t)$ increases. Therefore, this is a principal mechanism pushing the necking process. In the necking process, all modes of deviated motion may be excited by the presence of unpreventable, infinitesimal geometric or material imperfections or initial disturbances. However, only certain modes of motion having algebraically large values of Λ_p, which depend on the geometry, material properties and rate of stretching, may grow in amplitude. To determine quantitatively the necking process of an elastic–plastic cylinder, a numerical procedure and a computer program in double precision based on the foregoing approach have been developed. Eigenvalues and eigenmodes of motion are determined by a direct method and the integration of eqn (48) is accomplished by employing the sixth order Runge–Kutta method. At time increment $\Delta\tau$ of less than 1/50 of the shortest period of any oscillatory motion present in the deviated motion is employed. The power series by eqn (35) converges rapidly. Therefore, a limited number of terms may be sufficient to represent a solution. Accordingly, the number of distinct eigenmodes of motion will be dictated by the number of terms employed in the series representation. However, the eigenmodes of motion are determined by the extremum conditions of the functional Q and the respective values of Λ_p. Thus, the results of a limited series representation may cover sparsely a whole spectrum of deviated motion. Here, only first six terms of the series are employed in the computations. A number of cases with various combinations of other parameters have been considered and are described as follows.

7. NUMERICAL RESULTS

Numerical results have been obtained by the foregoing procedure for a number of cylinders. Each of the cylinders has initial imperfections leading to an initial state of the deviated motion respresented by a displacement profile which has the components of all eigenmodes of deformation of an equal amplitude with $\eta_p(0) = 0.00001$ and $\dot{\eta}_p(0) = 0$, $p = 1, \ldots, 6$. The typical results are shown in Figs. 1–6. All the cylinders referred to in Figs. 1–6 have the following material properties: $\bar{\eta} = 5$, $E_t^c/E = 0.1$ and various values of e_y such that, for each cylinder, the quasi-static bifurcation stress σ_c agrees with respective E_t^c and γ by eqn (50).

Figure 1 shows the histories of the amplitudes of a number of eigenmodes of motion of Case 1 having the parameters: $\gamma = 0.1$ and $v/c = 0.001$. The amplitudes of the first and sixth eigenmodes of motion grow with time as soon as the axial true stress exceeds the quasi-static bifurcation stress σ_c or $\tau > 0$. The first and sixth eigenmodes of motion predominate in the necking process while all other eigenmodes of motion are oscillatory and remain to be of small amplitudes in an order of that excited initially. The fifth eigenmode of motion is not shown as it has a much shorter period of oscillation than that of η_3 or η_4.

Figure 2 shows the amplitudes of a number of eigenmodes of motion of Case 2 having the parameters: $\gamma = 0.2$ and $v/c = 0.001$. Similar to that of Case 1, the first and sixth eigenmode of motion of this case also predominate, except that they grow at somewhat slower rates. It is noted that the product of $v\tau/c$ gives the value of the average axial strain in addition to that of the reference state. Thus, Cylinder 2 developed a neck of a certain amplitude at a slightly larger average axial strain than that of Cylinder 1. Figure 3 shows

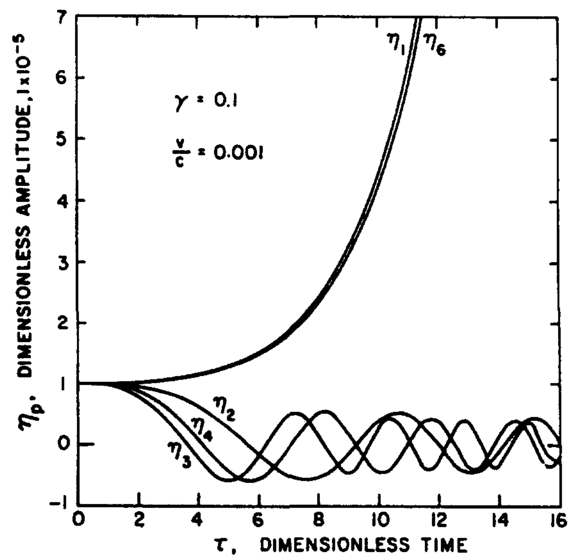

Fig. 1. Eigenmodes of motion of Case 1.

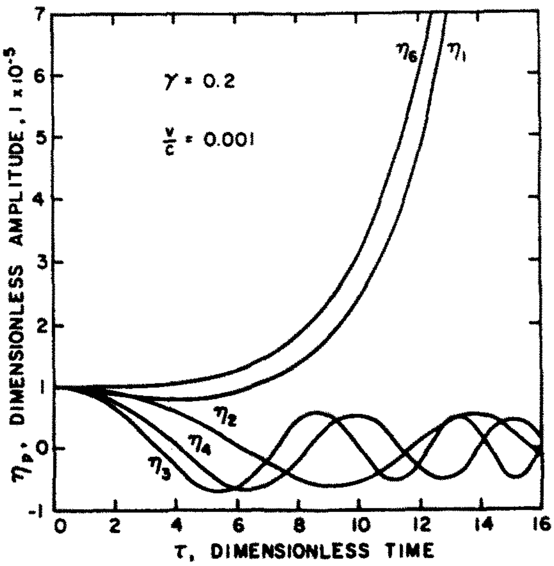

Fig. 2. Eigenmodes of motion of Case 2.

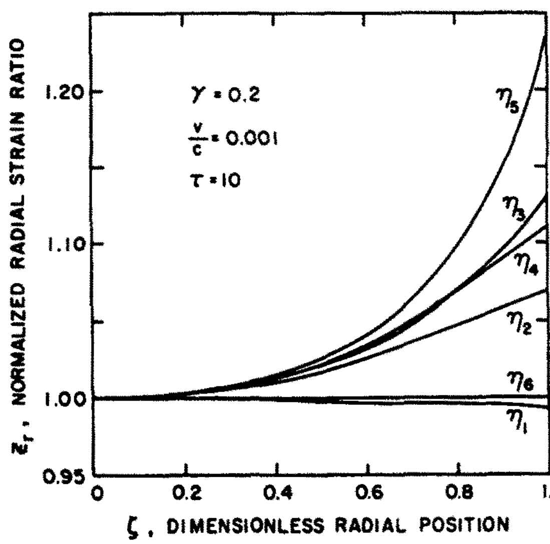

Fig. 3. Normalized radial strain profiles of Case 2.

the normalized radial strain profiles of the six eigenmodes of motion of Case 2, where $\bar{\epsilon}_r = \epsilon_r(\zeta)/\epsilon_r(0)$. For each of the cases considered, the eigenvalues and corresponding eigenmodes of motion are determined at each time increment. An eigenmode shape (or l_{mn}) does change with time but at an insignificant rate. The instantaneous eigenvalues are employed in integrating equation (48). The normalized radial strain profiles of the eigenmodes of motion of other cases have patterns and relative orders similar to this case.

Figure 4 shows the histories of the amplitudes of a number of eigenmodes of motion of Case 3 having the parameters: $\gamma = 0.4$ and $v/c = 0.001$. The second eigenmode of motion grows in amplitude continuously after $\tau > 0$. However, the first and sixth eigenmodes of motion grow at a later time. Cylinder 3 develops a neck of a certain amplitude at an average axial strain slightly larger than that of Cylinder 2.

Figure 5 shows the histories of the amplitudes of a number of eigenmodes of motion of Case 4 having the parametes: $\gamma = 0.1$ and $v/c = 0.0001$. The stretching velocity of this case is one tenth of that of Case 1. The first and sixth eigenmodes of motion also predominate and grow in amplitude soon after $\tau > 0$ at rates also much slower than that of Case 1. The

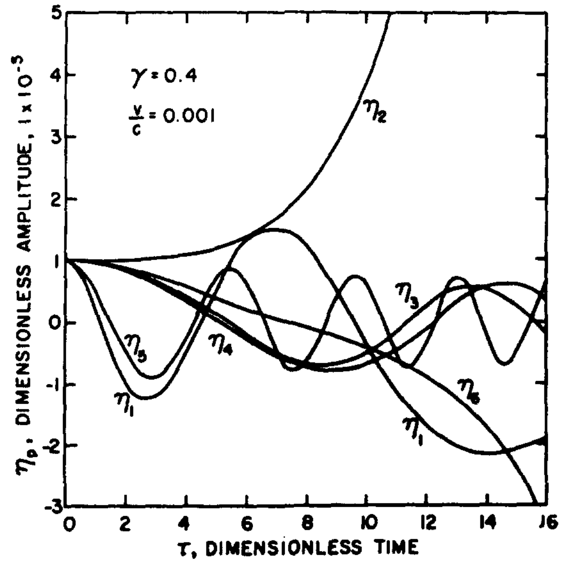

Fig. 4. Eigenmodes of motion of Case 3.

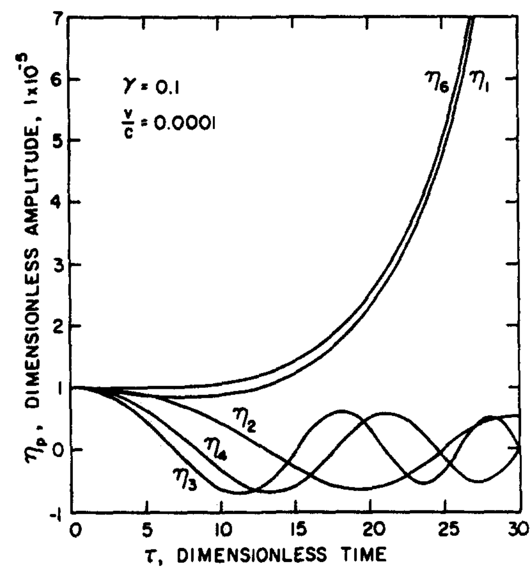

Fig. 5. Eigenmodes of motion of Case 4.

effects of stretching velocity on the rate of necking are illustrated by Fig. 6. Which shows a neck of a certain amplitude of a cylinder is developed at a shorter time by applying a greater stetching velocity. However, a neck of a certain amplitude such as that having $\epsilon_r(1) = -6 \times 10^{-5}$ occurs at a (e vs v/c) ratio of (0.00394 vs 0.0001), (0.00603 vs 0.0002), (0.0173 vs 0.001) or (0.078 vs 0.01) for each of the four cases shown, respectively. It is noted that the application of the current approach to the case of relatively high stretching velocity of $v/c = 0.01$ may be questionable. It is shown here to indicate a trend. The trend shows that a necking failure is developed in a cylinder at the quasi-static bifurcation true stress σ_c and the corresponding average axial strain when the stretching velocity approaches to zero. A number of other cases having material properties: $\bar{\eta} = 5$ and $E_t^c/E = 0.01$ have also been considered. Their necking patterns and processes are similar to the series presented here except they have slower necking rates.

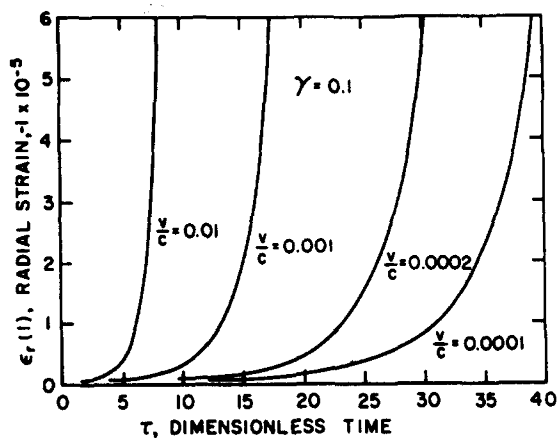

Fig. 6. Rates of necking at various stretching velocities.

8. CONCLUDING REMARKS

The computer program leading to the numerical results illustrated in paper has certain limitations. The program has been prepared for cases where the deviated motion is relatively small, the radius to length ratio γ is relatively small and the wave effects are negligible. With additional modifications, the concept and approach presented herein may still be applicable to cases beyond the limitations. The results presented herein do indicate that the problem of dynamic necking of an elastic–plastic cylinder may be treated as an initial-value-eigenvalue problem. The eigenmodes of motion can be obtained from a functional Q. The eigenmodes of motion depend on the geometry, material properties and stretching velocity of the cylinder. The initiation of an eigenmode of motion does require the presence of that particular mode of initial imperfection or disturbance, no matter how small it is. When all eigenmodes of motion are initially excited, only certain eigenmodes of motion predominate at a time depending on the characteristics of the functional Q which describes the intrinsic interactions between the undisturbed and disturbed motions of the cylinder.

It is noted that the current approach is applicable to systems subjected to generalized forces which are configuration-dependent. For stability problems involving velocity-dependent forces, additional considerations are required.

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